# THE WEAK ISOVARIANT BORSUK-ULAM THEOREM FOR COMPACT LIE GROUPS

IKUMITSU NAGASAKI


## Abstract

As a variant of the Borsuk-Ulam theorem, A. G. Wasserman [3] studied the following problem:

**Problem.** *If (continuous) $G$-isovariant map $f : V \to W$ between $G$-representation spaces exists, does the inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

*hold?*

It is known that the inequality holds if $G$ is solvable, however this problem is unsolved in general.

We investigate a weaker version of the above problem and obtain the following.

**Theorem 1.** *For an arbitrary compact Lie group $G$, there exists a* (*weakly*) *monotone increasing function $\varphi_G : \mathbb{N}_0 \to \mathbb{N}_0$* ($\mathbb{N}_0$ *the nonnegative integers*) *diverging to $\infty$ with the following property:*

- *If there exists a $G$-isovariant map $f : V \to W$, then the inequality*

  $$\varphi_G(\dim V - \dim V^G) \leq \dim W - W^G.$$

  *holds.*

We also study an analogous problem for spheres with semilinear actions. We call a closed manifold $M$ a semilinear $G$-sphere if for every $H \leq G$, $M^H$ is a (homotopy) sphere or empty. Though it seems that a semilinear action resembles to a linear action in fixed point sets, actually several phenomena different from linear cases occur. As an example we shall illustrate the (weak) isovariant Borsuk-Ulam theorem in the family $\mathcal{S}_G$ of semilinear $G$-spheres.

Definition**.** We say that $G$ has the WIB-property in $\mathcal{S}_G$ if there exists a (weakly) monotone increasing function $\varphi_G : \mathbb{N}_0 \to \mathbb{N}_0$ diverging to $\infty$ with the following property:

- For any $M$, $N \in \mathcal{F}_G$, if there is a $G$-isovariant map $f : M \to N$, then

$$\varphi_G(\dim M - \dim M^G) \leq \dim N - \dim N^G.$$

If we can take $\varphi_G$ to be the identity map, we say that $G$ has the IB-property in $\mathcal{S}_G$.

In this case we show:

**Theorem 2.** *The following statements are equivalent.*

(1) *$G$ has the IB-property in $\mathcal{S}_G$.*
(2) *$G$ has the WIB-property in $\mathcal{S}_G$.*
(3) *$G$ is solvable.*

Namely, in the family of semilinear $G$-spheres, the (weak) isovariant Borsuk-Ulam theorem holds if and only if $G$ is solvable.

## References

[1] I. Nagasaki, *The weak isovariant Borsuk-Ulam theorem for compact Lie groups*, to appear in Arch. Math.
[2] R. Oliver, *Smooth compact Lie group actions on disks*, Math. Z. **149** (1976), 71–96.
[3] A. G. Wasserman, *Isovariant maps and the Borsuk-Ulam theorem*, Topology Appl. **38** (1991), 155–161.

Department of Mathematics, Graduate School of Science, Osaka University Toyonaka, Osaka 560-0043, Japan
*E-mail address*: nagasaki@math.sci.osaka-u.ac.jp