A decomposition of equivariant stable homotopy classes and a computation of the first equivariant stem.

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Abstract: The stable equivariant homotopy classes of equivariant (pointed) maps between unit spheres of orthogonal representations, called stable stems, $\pi_*^{G\,\mathrm{st}}$, are of an interest. It is quite easy to show that the negative G-stems are zero, that is $\pi_k^{G\,\mathrm{st}} = 0$, if $, k < 0$. In 1970, Segal [13], stated that for any finite group G, $\pi_0^{G\,\mathrm{st}} \cong A(G)$, where $A(G)$ is the Burnside ring of G. This result had been proved also by Kosniowski [11], and independently by Rubinsztein [12] with a gap that was filled later by Dancer. T. tom Dieck [3] proved the same result for a general compact Lie group G, giving a convenient definition of the Burnside ring $A(G)$ for this case. We give a full computation of the first equivariant (stable) stem for G, $\pi_1^{G\,\mathrm{st}} = \mathrm{colim}_V\,[\mathbb{S}^{V+1}, \mathbb{S}^V]_G$, where V varies along a cofinal set of orthogonal G-representations.

Theorem 1. *Let G be any compact Lie group. There is a direct sum decomposition ranking the set Or_G of conjugacy classes of subgroups of G :*

$$\pi_1^{G\,\mathrm{st}} = \bigoplus_{\substack{(H)\in\mathrm{Or}_G\\ \dim W(H)\le 1}} \Pi_1(H)\,, \quad \textit{where } W(H) \textit{ is the Weyl group of } H\,.$$

Here, if $\dim W(H) = 0$,

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}}\,,$$

where $W(H)_{\mathrm{ab}}$ is the abelianization of $W(H)$, and, if $\dim W(H) = 1$,

$$\Pi_1(H) \cong \begin{cases} \mathbb{Z} & W(H) \textit{ is biorientable,} \\ \mathbb{Z}_2 & \textit{if } W(H) \textit{ is not biorientable.} \end{cases}$$

We base on the previous results on equivariant homotopy groups as of Ize et al., who have made many computations of $\pi_*^{G\,\mathrm{st}}$ when G is abelian ([5, 6, 7, 8, 9, 10]). Balanov and Krawcewicz [1] showed for a general compact Lie group G that there is a direct sum decomposition

$$\pi_k^{G\,\mathrm{st}} \cong \bigoplus_{(H)} \Pi_k(H)\,,$$

where $\Pi_k(H)$ denotes the subgroup of $\pi_k^{G\,\mathrm{st}}$ corresponding to the isotropy type (H), for a subgroup $H \subset G$; the sum ranks over all (H) such that $\dim W(H) \le k$. Following computations made in [4], where a construction of the equivariant degree is given, one obtains that if $\dim W(H) = k$, then $\Pi_1(H) \cong \mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether $W(H)$ is biorientable or not. On the other hand, in the treatment of $\pi_1^{G\,\mathrm{st}}$ made in [1] it was shown that for $\Pi_1(H) \subset \pi_1^{G\,\mathrm{st}}$, with $\dim W(H) = 0$, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Pi_1(H) \longrightarrow W(H)_{\mathrm{ab}} \longrightarrow 0\,. \tag{0.1}$$

In [2], Balanov, Krawcewicz, and Steinlein, using results of Ize and purely algebraic arguments, proved that this sequence splits when G is abelian. By a natural geometric argument we show that this exact sequence splits always, which completes the computation of the first equivariant steam.

Moreover, for any compact Lie group G, we give a decomposition of the group $\{X,Y\}_G^k$ of (unpointed) stable G-homotopy classes as a direct sum of subgroups of fixed orbit types. This is done by interpreting the G-homotopy classes in terms of the generalized fixed point transfer and making use of conormal maps.

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