THE TOPOLOGY OF THE BANACH-MAZUR COMPACTA

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In his 1932 book *Théorie des Opérations Linéaires*, S. Banach introduced the space of isometry classes $[X]$, of n-dimensional Banach spaces equipped with the famous Banach-Mazur metric:

$$d([X],[Y]) = \ ln\ inf\big\{\|T\|\cdot\|T^{-1}\| \ \big|\ T : X \to Y \text{ is a linear isomorphism}\big\}.$$

These spaces are now denoted by $BM(n)$ and called the Banach-Mazur compacta. A famous problem of A. Pelczyński from [1, Problem 899] asks whether $BM(n)$, $n \geq 2$ is a Hilbert cube? We answered this question in [2] in the negative for $n = 2$.

In this talk we shall present some resent results and open problems related to these interesting objects; most of them are contained in [2] and [3].

Let $\mathcal{B}(n)$ be the hyperspace of all centrally symmetric, compact, convex bodies of n-dimensional Euclidean space, endowed with the Hausdorff metric topology.

It follows easily from the Minkowski functional argument that the Banach-Mazur compactum $BM(n)$ is homeomorphic to the orbit space $\mathcal{B}(n)/GL(n)$ of the natural action of the general linear group $GL(n)$ on $\mathcal{B}(n)$.

Let $L(n)$ be the subset of all compact convex bodies $A \in \mathcal{B}(n)$ such that the ordinary Euclidean unit ball B^n is the ellipsoid of minimal volume containing A. Let $L_0(n) = L(n) \setminus \{B^n\}$.

The following holds true:

1. $\mathcal{B}(n)$ is a proper $GL(n)$-space.
2. $L(n)$ is a global $O(n)$-slice for $\mathcal{B}(n)$; consequently $BM(n)$=$J(n)/O(n)$.
3. For each closed subgroup $H \subset O(n)$ that acts nontransitively on the sphere S^{n-1}, the fixed point set $L(n)[H] = \{A \in L(n) |\ hA = A,\ \forall h \in H\}$, as well as the H-orbit space $L(n)/H$, is a Hilbert cube. In particular, $L(n)$ is a Hilbert cube.
4. For each closed subgroup $H \subset O(n)$, the H-orbit space $L_0(n)/H$ is a Hilbert cube manifold.
5. $\mathcal{B}(n)$ is homeomorphic to $Q \times \mathbb{R}^{\frac{n(n+1)}{2}}$, where Q is the Hilbert cube.
6. The $SO(2)$-orbit space $L_0(2)/SO(2)$ is an Eilenberg-MacLane space of type $\mathbf{K}(\mathbb{Q}, 2)$. The induced action of the cyclic group $\mathbb{Z}_2 = O(2)/SO(2)$ on $L_0(2)/SO(2)$ has a noncontractible orbit space $BM_0(2) = \frac{L_0(2)/SO(2)}{\mathbb{Z}_2}$ and $BM(2) = \frac{L(2)/SO(2)}{\mathbb{Z}_2}$.

7. Let $(H_1), (H_2), \ldots$ be the sequence of all orbit types occuring in the $O(n)$-space $L_0(n)$. Let $\Pi(n)$ be the product $\prod_{i=1}^{\infty}\big(con(O(n)/H_i)\big)^{\infty}$ equipped with the diagonal $O(n)$-action and let $\Pi_0(n)$ denote the complement of the unique $O(n)$-fixed point in $\Pi(n)$.

We prove that $\Pi_0(n)/O(n)$ is a Hilbert cube manifold and $BM(n)$ is just homeomorphic to the orbit space $\Pi(n)/O(n)$.

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8. Let $\Delta(2) = \prod_{k=1}^{\infty} D_k$, where D_k is the unit disk of $\mathbb{R}^2$ endowed with the natural $O(2)$-action which kernel is the cyclic subgroup $\mathbb{Z}_k$ of $O(2)$. Let $\Delta_0(2)$ denote the complement of the unique $O(2)$-fixed point in $\Delta(2)$.

Then $L_0(2)/O(2) \cong \Delta_0(2)/O(2)$; consequently $BM(2) \cong \Delta(2)/O(2)$.

Other related results and open problems will also be discussed.

References

1. J. E. West, *Open Problems in Infinite-dimensional topology*, in: Open Problems in Topology (ed. by J.van Mill and G. Reed), 524–586, North Holland, Amsterdam-New York-Oxford-Tokyo, 1990.
2. S. A. Antonyan, *The topology of the Banach-Mazur compactum*, Fund. Math. **166, no. 3** (2000), 209-232.
3. S. A. Antonyan, *West's problem on equivariant hyperspaces and Banach-Mazur compacta*, Trans. Amer. Math. Soc. **335, No. 8** (2003), 3379-3404.