# The equivariant diffeomorphism groups of $G$-manifolds and the smooth structure of the orbit spaces

Let $M$ be a connected smooth closed manifold and let $D(M)$ denote the group of $C^\infty$-diffeomorphisms of $M$ which are isotopic to the identity. Herman and Thurston proved that $D(M)$ is a perfect group. There are many results in this direction on the group of diffeomorphisms of $M$.

We shall consider the case when a compact Lie group $G$ acts smoothly on a manifold $M$. Let $D_G(M)$ denote the equivariant diffeomorphism group of a $G$-manifold $M$ whose elements are $G$-isotopic to the identity. Then $D_G(M)$ is a perfect group when $M$ is a principal $G$-manifold or a $G$-manifold with one orbit type. If $M$ has more than two orbit types, then $D_G(M)$ is not perfect in general. We shall give a brief survey of the results on the first homology group $H_1(D_G(M)) = D_G(M)/[D_G(M), D_G(M)]$.

Relevant to the structure of the group $D_G(M)$, we shall also discuss the smooth structure of the orbit space $M/G$ of $M$ which was introduced by Bierstone and Schwartz. They proved that smooth isotopies of $M/G$ are covered by equivariant smooth isotopies of $M$, which implies that the study of the group $D(M/G)$ gives the fundamental information to $D_G(M)$. From the result of Strub, if $M$ is a representation space of a finite subgroup $G$ of the orthogonal group, then the smooth structure of the orbit space determines the equivalence class of the representation space $M$. It follows that the local structure of a $G$-manifold is determined by the smooth structure of its orbit space.

We shall refer to some other results relating to the above subject.